Consider the IBVP:

$$\begin{cases} u_t = u_{xx}, & 0<x<1,\ t>0 & (13.1)\\ u(x,0) = \phi(x), & 0\le x\le 1 & (13.2)\\ u(0,t) = u(1,t) = 0 & & (13.3)\end{cases}$$

A qualitative information of the behavior of the solution of (13) can be obtained by analyzing the L^2-norm of the soln without solving the problem:

In fact,

$$\|u(\cdot,t)\|_2 \equiv \left(\int_0^1 u^2(x,t)\,dx\right)^{1/2}$$

First multiply by u and integrate $\int_0^1 dx$

$$\int_0^1 (u\,u_t)(x,t)\,dx = \int_0^1 (u\,u_{xx})(x,t)\,dx$$

$$(u\,u_x)_x = u_x u_x + u\,u_{xx}$$

Assuming sufficient smoothness

$$\int_0^1 \frac{d}{dt}\left(\frac{u^2}{2}\right)dx = -\int_0^1 u_x^2\,dx + \int_0^1 (u\,u_x)_x\,dx$$

thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx = u\,u_x\Big|_0^1 - \int_0^1 u_x^2\,dx$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u(x,t)\|_2^2 = u(1,t)\underbrace{u_x(1,t)}_{\overset{B.C.}{=0}} - u(0,t)\underbrace{u_x(0,t)}_{\overset{B.C.}{=0}} - \int_0^1 u_x^2(x,t)\,dx$$

Then,

$$\frac{d}{dt}\|u(x,t)\|_2^2 = -2\int_0^1 u_x^2(x,t)\,dx \leq 0$$

By defining,

$$E(t) \equiv \int_0^1 u^2(x,t)\,dx = \|u(x,t)\|_2^2 \qquad (14.1)$$

leads to

$$\frac{dE}{dt}(t) \leq 0 \qquad (14.2)$$

This implies that the function $E(t)$ is nonincreasing.

This analysis also serves to prove uniqueness of solutions for the BVP:

$$\begin{cases} u_t = u_{xx} & (15.1) \\ u(x,0) = \phi(x) & (15.2) \\ u(0,t) = h(t), \quad u(1,t) = g(t). & (15.3) \end{cases}$$

In fact, if $u_1(x,t)$ and $u_2(x,t)$ are solns. of (15.1)-(15.3).

then $\boxed{w(x,t) = u_1(x,t) - u_2(x,t)}$ is also a Solution of

$$\begin{cases} w_t = w_{xx} & (15.4) \\ w(x,0) = 0 & (15.5) \\ w(0,t) = 0, \quad w(1,t) = 0 & (15.6) \end{cases}$$

why?

And from previous analysis

$$\frac{d}{dt} E(t) = \frac{d}{dt}\int_0^1 w^2(x,t)\,dx \le 0 \qquad (15.7)$$

Also, for (15.4)-(15.6), $E(0) = \int_0^1 w^2(x,0)\,dx = 0$ (15.8)

and $E(t) = \int_0^1 w^2(x,t)\,dx \geq 0$ (15.9)

The results (15.7)-(15.9) plus the continuity of $E(t)$

leads to

$$E(t) \equiv 0$$

or $$\int_0^1 w^2(x,t)\,dx = 0, \text{ for all } t \Rightarrow w(x,t) = 0$$

$$\Rightarrow \quad u_1(x,t) = u_2(x,t). \qquad (16.1)$$

(16.1) means that if there is soln. of (15.1)-(15.3) solution is unique.

For the original problem (13.1) – (13.3)

If there is t^* s.t. $\frac{dE}{dt}(t^*) = \frac{d}{dt}\int_0^1 u^2(x,t^*)\,dx = 0$

$$\Rightarrow \quad -\int_0^1 u_x^2(x,t^*)\,dx = 0 \Rightarrow u_x(x,t^*) = 0$$

$$\Rightarrow \quad u(x,t^*) = C, \text{ but } \overset{\text{BCs.}}{u(0,t^*) = 0 = u(1,t^*)}$$

$$\Rightarrow \quad u(x,t^*) = 0$$

$$\Rightarrow \quad E(t^*) = \int_0^1 u^2(x,t^*)\,dx = 0$$

Also, $\frac{dE}{dt}(t) \leq 0$ and $E(t) \geq 0$, for all t.

then $E(t) = 0$, $t \geq t^* \Leftrightarrow u(x,t) = 0$, $t \geq t^*$

Initially,

$$E(0) = \int_0^1 u^2(x,0)\,dx = \int_0^1 \phi^2(x)\,dx \neq 0 \quad \text{if } \phi(x) \neq 0$$

Also

$$\frac{dE}{dt}(0) = -2\int_0^1 u_x^2(x,0)\,dx = -2\int_0^1 (\phi'(x))^2\,dx < 0,$$

If $\phi(x) \not\equiv$ Constant

Also, the only possibility for $\phi \equiv$ Const. is if $\phi(x) \equiv 0$ in $[0,1]$, because $\phi(0) = u(0,t) = 0 = u(1,t) = \phi(1)$